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NOTE

Extensions of the Poisson Summation Formula

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The classical Poisson summation formula (1.1) and the corresponding distributional formula (1.2) have found extensive applications in various scientific fields. However, they are not universally valid. For instance, if $\phi(x)$ is a smooth function, the left-hand side of (1.1) is generally divergent. Even when both sides of (1.1) converge absolutely, they may do so to different numbers. Indeed, in Example 3 we are faced with the embarrassing situation where the series on the left-hand side of (1.1) converges for $\operatorname{Re} s > 1$ while that on the right-hand side converges only for $\operatorname{Re} s < 0$. Our aim is to extend formulas (1.1) and (1.2) with the help of some new results in distributional theory. For instance, the evaluation of the distribution with zero mean as given by (3.1) at a test function $\phi(x)$ yields the relation $\sum_{-\infty}^{\infty} \phi(k) - \int_{-\infty}^{\infty} \phi(x)\,dx$. Both the series and the integral in this expression are generally divergent. The concept of the Cesàro limit is then used to obtain the finite difference of these two terms. Thereafter we extend the analysis to higher dimensions. Various innovative examples are presented to illustrate these concepts.

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1. INTRODUCTION

The Poisson summation formula

$$\sum_{k=-\infty}^{\infty} \phi(k) = \sum_{k=-\infty}^{\infty} \hat{\phi}(2\pi k), \qquad k \in \mathbb{Z}, \tag{1.1}$$

is a very useful tool in many branches of mathematics. It relates sums involving a function ϕ and its Fourier transform $\hat{\phi}(u) = \int_{-\infty}^{\infty} e^{iux}\phi(x)\,dx$ and possesses great simplicity and beauty.

However, as is clear, it will be valid only if ϕ and its Fourier transform are regular enough and only if both series converge.

The purpose of this note is to present a rather simple variant of (1.1) that holds for a very large class of functions ϕ. This variant applies even if one or both series in (1.1) are divergent.

Introducing a suitable parameter in (1.1), we obtain the related formula

$$\sum_{k=-\infty}^{\infty} f(k\lambda) = \frac{1}{|\lambda|} \sum_{k=-\infty}^{\infty} \hat{f}\left(\frac{2\pi k}{\lambda}\right), \tag{1.2}$$

which is (1.1) applied to $f(\lambda x)$. Formula (1.2) can then be interpreted in the distributional sense with respect to λ. This idea was employed by Ortner and Wagner [18], who gave an elegant solution to a problem of Henkel and Weston [14]. Observe, however, that (1.2) will hold for a very restricted class of distributions and, certainly, cannot be applied to all $f \in \mathscr{S}'$, the space of tempered distributions. We also give a variant of the distributional Poisson formula (1.2) and show that this variant can be applied to *all* tempered distributions.

The pointwise summation formula (1.1) and the distributional summation formula (1.2) both have multidimensional versions. We show that our variant applies in this context as well.

In Section 2 we give some recent results on distribution theory that will be needed in the sequel. The pointwise variants in one variable are considered in Section 3, while the distributional formulas are given in Section 4. Finally, Section 5 gives the extensions to several variables. The results are illustrated with several examples.

2. PRELIMINARIES

In this section we discuss some results from the theory of distributions that will be needed in our analysis. For the basic facts on the theory of distributions, in particular, on the spaces $\mathscr{D}'$, $\mathscr{E}'$, and $\mathscr{S}'$, we refer to the textbooks [9, 15, 21].

The leading role in this article is played by the space of test functions $\mathscr{K}$ and its dual space $\mathscr{K}'$ [9, Chap. 2]. The space $\mathscr{K}$ was introduced in [12] and is sometimes referred as the space of GLS symbols. It is constructed as follows. Let $q \in \mathbb{R}$. Then a function ϕ defined in $\mathbb{R}^d$ belongs to $\mathscr{K}_q$ if ϕ is smooth and if for each multiindex $\boldsymbol{\alpha} \in \mathbb{N}^d$ it satisfies $\mathbf{D}^{\boldsymbol{\alpha}}\phi(\mathbf{x}) = O(|\mathbf{x}|^{q-|\boldsymbol{\alpha}|})$ as $|\mathbf{x}| \to \infty$. Here we use the standard notation $\mathbf{D}^{\boldsymbol{\alpha}} = \partial^{|\boldsymbol{\alpha}|}/\partial x_1^{\alpha_1} \cdots \partial x_d^{\alpha d}$, $|\boldsymbol{\alpha}| = \alpha_1 + \cdots + \alpha_d$, $\boldsymbol{\alpha}! = \alpha_1! \cdots \alpha_d!$. A topology is obtained by considering the family of seminorms

$$\|\phi\|_{q,\boldsymbol{\alpha}} = \sup_{\mathbf{x}\in\mathbb{R}^d} \rho_q(|\boldsymbol{\alpha}|)|\mathbf{D}^{\boldsymbol{\alpha}}\phi(\mathbf{x})|, \tag{2.1}$$

where $\rho_q(r) = 1,\ 0 \le r \le 1,\ \rho_q(R) = r^{-q},\ r > 1$. The space $\mathscr{K}$ is the inductive limit of the spaces $\mathscr{K}_q$ as $q \to \infty$. Observe that every polynomial belongs to $\mathscr{K}$.

The dual space $\mathscr{K}'$ is fundamental in the theory of distributional asymptotic expansions [8, 9, 19, 23]. Indeed, it has been found recently [8] that each element f of $\mathscr{K}'(\mathbb{R}^d)$ is ‘‘distributionally small’’ at infinity in the sense that f satisfies the *moment asymptotic expansion*

$$f(\lambda\mathbf{x}) \sim \sum_{\substack{|\boldsymbol{\alpha}|=0\\ \boldsymbol{\alpha}\in\mathbb{N}^d}}^{\infty} \frac{(-1)^{|\boldsymbol{\alpha}|}\mu_{\boldsymbol{\alpha}}\mathbf{D}^{\boldsymbol{\alpha}}\,\delta(\mathbf{x})}{\boldsymbol{\alpha}!\,\lambda^{|\boldsymbol{\alpha}|+d}} \qquad \text{as } \lambda \to \infty, \tag{2.2}$$

where $\mu_{\boldsymbol{\alpha}} = \langle f(\mathbf{x}), \mathbf{x}^{\boldsymbol{\alpha}}\rangle$ are the moments. The converse also holds, that is, that any distributionally small generalized function belongs to $\mathscr{K}'$ was proved very recently [5].

Another very important result from [5] is that in the one-dimensional case any evaluation $\langle f(x), \phi(x)\rangle$ of a distribution $f \in \mathscr{K}'(\mathbb{R})$ at a test function $\phi \in \mathscr{K}(\mathbb{R})$ can be computed as a limit in the Cesàro sense. In this article we extend this result to evaluation in $\mathscr{K}'(\mathbb{R}^d) \times \mathscr{K}(\mathbb{R}^d)$, see Section 5.

The idea of interpreting distributional evaluations as Cesàro limits is best explained by using the order symbols $O(x^{\alpha})$ and $o(x^{\alpha})$ in the Cesàro sense. If $f \in \mathscr{D}'(\mathbb{R})$ and $\alpha \in \mathbb{R} \setminus \{-1, -2, -3, \dots\}$ we say that $f(x) = O(x^{\alpha})$ as $x \to \infty$ in the Cesàro sense and write

$$f(x) = O(x^{\alpha}) \qquad (C) \qquad x \to \infty, \tag{2.3}$$

if there exists $N \in \mathbb{N}$ such that each primitive F of order N of f (i.e., $F^{(N)} = f$) is an ordinary function for large arguments and satisfies the ordinary order relation

$$F(x) = p(x) + O(x^{\alpha+N}) \qquad \text{as } x \to \infty \tag{2.4}$$

for a suitable polynomial of degree $N - 1$ at the most. A similar definition applies to the o symbol.

We say that a distribution $f \in \mathscr{D}'$ has the limit L in the Cesàro sense as $x \to \infty$ and write $\lim_{x \to \infty} f(x) = L \ (C)$ if $f(x) = L + o(1) \ (C)$ as $x \to \infty$.

Distributional evaluations are treated as follows. Suppose first that $\operatorname{supp} f$ is bounded on the left and consider the evaluation $\langle f(x), \phi(x) \rangle$. Let $g(x)$ be the first-order primitive of $\phi(x) f(x)$ with support bounded on the left. We say that the evaluation $\langle f(x), \phi(x) \rangle$ exists in the Cesàro sense and equals L and write

$$\langle f(x), \phi(x) \rangle = L \qquad (C) \tag{2.5}$$

if $\lim_{x \to \infty} g(x) = L \ (C)$. The case when the support of f is bounded on the right is similar and the general case is handled by writing $f = f_1 + f_2$ with $\operatorname{supp} f_1$ bounded on the left and $\operatorname{supp} f_2$ bounded on the right. It is easy to see that the Cesàro limit of the evaluation is independent of the decomposition.

The elements of $\mathscr{K}'$ admit another interesting characterization [5]: a distribution $f \in \mathscr{D}'(\mathbb{R})$ belongs to $\mathscr{K}'(\mathbb{R})$ if and only if $f(x) = O(|x|^{-|\alpha|}) \ (C)$ as $x \to \pm\infty$ for each $\alpha > 0$, that is, if and only if $f(x) = O(|x|^{-\infty}) \ (C)$ as $x \to \pm\infty$. In a sense the distributions in the space $\mathscr{K}'$ are like the test functions of $\mathscr{S}$, because both decay very fast at ∞, the former in the Cesàro sense while the later in the ordinary sense. There is also an analogy between $\mathscr{K}$ and the space of tempered distributions $\mathscr{S}'$.

We shall also need the behavior of the Fourier transform on the spaces $\mathscr{K}$ and $\mathscr{K}'$. We denote by $\hat{f}(\mathbf{u})$ or $\mathscr{F}\{f(\mathbf{x}); \mathbf{u}\}$ the Fourier transform of a tempered distribution $f \in \mathscr{S}'(\mathbb{R}^d)$, so that $\mathscr{F}\{f(\mathbf{x}); \mathbf{u}\} = \int_{\mathbb{R}^d} \exp(i\mathbf{x} \cdot \mathbf{u}) f(\mathbf{x})\, d\mathbf{x}$ if $f \in L'(\mathbb{R}^d)$.

If $\phi \in \mathscr{K}(\mathbb{R}^d)$, then its Fourier transform is smooth in $\mathbb{R}^d \setminus \{0\}$ and is of rapid decay at ∞, that is, $\hat{\phi}(\mathbf{u}) = O(|\mathbf{u}|^{-\infty})$ as $|\mathbf{u}| \to \infty$. From the results of [5] in one dimension and from the results of Section 5 in $\mathbb{R}^d$, we find that if $\mathbf{u} \neq 0$, then $\hat{\phi}(\mathbf{u})$ is equal to the Cesàro value of the generally divergent integral $\int_{\mathbb{R}^d} \exp(i\mathbf{x} \cdot \mathbf{u}) \phi(\mathbf{x})\, d\mathbf{x}$.

On the other hand, a distribution $f \in \mathscr{S}'(\mathbf{u})$ belongs to $\mathscr{K}'(\mathbb{R})$ if and only if the distributional point values of all derivatives of the Fourier transform at the origin, $\hat{h}^{(n)}(0)$, $n = 0, 1, 2, \ldots,$ exist. Recall [16] that if $h \in \mathscr{D}'$, then γ is called the distributional point value of $h(x)$ at $x = x_0$, written $h(x_0) = \gamma$ in $\mathscr{D}'$, if $\lim_{\epsilon \to 0} h(x_0 + \epsilon x) = \gamma$ in the topology of the space $\mathscr{D}'$.

3. AN EXTENDED FORMULA

If ϕ is a smooth function of the space $\mathscr{K}(\mathbb{R})$, then the series $\sum_{k=-\infty}^{\infty} \phi(k)$ is generally divergent. Actually, in general, the series $\sum_{k=-\infty}^{\infty} \phi(k)$ is not even Cesàro or Abel summable. One can take ϕ to be a polynomial, for instance.

However, as we now show, a simple modification of the series $\sum_{k=-\infty}^{\infty} \phi(k)$ has a finite value. Indeed, any periodic distribution with zero mean belongs to $\mathscr{K}'$. In particular, the distribution

$$\sigma(x) = \sum_{k=-\infty}^{\infty} \delta(x-k) - 1 \tag{3.1}$$

belongs to $\mathscr{K}'$. Thus, if $\phi \in \mathscr{K}$ the quantity $\langle \sigma(x), \phi(x) \rangle$ gives a value to $\sum_{k=-\infty}^{\infty} \phi(k) - \int_{-\infty}^{\infty} \phi(x)\, dx$, the difference of a generally divergent series and a generally divergent integral. Actually, it can be shown that any evaluation $\langle f(x), \psi(x) \rangle$ with $f \in \mathscr{K}'$ and $\psi \in \mathscr{K}$ can be computed in the Cesàro sense. Therefore,

$$\lim_{A\to\infty} \left(\sum_{|k|\le A} \phi(k) - \int_{-A}^{A} \phi(x)\, dx \right) = \langle \sigma(x), \phi(x) \rangle \qquad (C). \tag{3.2}$$

If g is a periodic distribution of period p with zero mean, then its Fourier series $\sum_{k=-\infty}^{\infty} a_k \exp(2\pi ikx/p)$, which certainly converges in $\mathscr{S}'$, also converges in $\mathscr{K}'$. In particular, if $g = \sigma$, then $p = 1$ and $a_k = 1$, $k \neq 0$, so that

$$\sigma(x) = \sideset{}{'}\sum_{k=-\infty}^{\infty} \exp(2\pi ikx) \tag{3.3}$$

in the topology of $\mathscr{K}'$. Here the prime means that the term corresponding to $k = 0$ is omitted. Evaluating at a test function, we thus obtain

$$\begin{aligned} \langle \sigma(x), \phi(x) \rangle &= \sideset{}{'}\sum_{k=-\infty}^{\infty} \langle \exp(2\pi ikx), \phi(x) \rangle \\ &= \sideset{}{'}\sum_{k=-\infty}^{\infty} \hat{\phi}(2\pi k), \end{aligned}$$

and, using (3.2),

$$\sum_{k=-\infty}^{\infty} \phi(k) - \int_{-\infty}^{\infty} \phi(x)\, dx = \sideset{}{'}\sum_{k=-\infty}^{\infty} \hat{\phi}(2\pi k) \qquad (C). \tag{3.4}$$

Observe that the expression on the left-hand side is to be evaluated in the Cesàro sense, while the sum on the right-hand side is a rapidly convergent series because $\hat{\phi}(u)$ is smooth for $u \neq 0$ and of rapid decay at ∞.

Equation (3.4) is our first extension of the Poisson summation formula. It is obtained from the standard form (1.1) by transposing to the left-hand side the term $\hat{\phi}(0)$, which equals $\int_{-\infty}^{\infty} \phi(x)\, dx$ in case of convergence of this integral. This apparently trivial operation on (1.1) has far-reaching consequences, however. Indeed, (3.4) applies to functions of $\mathscr{K}$, and $\mathscr{K}$ is a *very large class*; on the other hand, (1.1) applies to a rather restrictive class of functions.

Let us give some illustrations.

EXAMPLE 1. Let us take $\phi(x) = 1$. Then $\hat{\phi}(u) = 2\pi\delta(u)$, so that $\hat{\phi}(u) = 0$ for $u \neq 0$. Thus, the right-hand side of (3.4), $\sum_{k=-\infty}^{\prime\infty}\hat{\phi}(2\pi k)$, vanishes. Hence, the left-hand side vanishes as well:

$$\sum_{k=-\infty}^{\infty} 1 - \int_{-\infty}^{\infty} dx = 0 \qquad (C)$$

in the sense that

$$\lim_{A\to\infty}\left(\sum_{k=-[A]}^{[A]} 1 - \int_{-A}^{A} dx\right) = 0 \qquad (C), \tag{3.5}$$

where $[A]$ is the greatest integer less than or equal to A.

Formula (3.5) can be easily verified, since

$$\begin{aligned}\sum_{k=-[A]}^{[A]} 1 - \int_{-A}^{A} dx &= 2[A] + 1 - 2A\\ &= 1 - 2(A - [A])\\ &= 1 - 2\{A\},\end{aligned}$$

where $\{A\} = A - [A]$ is the fractional part of A. But $\{A\}$ is a periodic function of A, with mean $1/2$ and thus [5, 8]

$$\{A\} = 1/2 + o(1) \qquad (C) \qquad \text{as } A \to \infty, \tag{3.6}$$

so that (3.5) follows.

EXAMPLE 2. Let us now consider the extended Poisson formula (3.4) when $\phi(x) = x^n$, $n = 1, 2, 3, \dots$. In this case $\hat{\phi}(u) = (-i)^n 2\pi\delta^{(n)}(u)$ and again the right-hand side of (3.4) vanishes. Therefore,

$$\sum_{k=-\infty}^{\infty} k^n - \int_{-\infty}^{\infty} x^n\, dx = 0 \qquad (C). \tag{3.7}$$

This is trivial when n is odd. When n is even, $n \geq 2$, it yields

$$\sum_{k=1}^{[A]} k^n - \int_0^A x^n\, dx = o(1) \qquad (C) \qquad \text{as } A \to \infty. \tag{3.8}$$

But [9, p. 35]

$$\sum_{k=M+1}^{N} k^n = \frac{1}{n+1}\big(B_{n+1}(N) - B_{n+1}(M)\big), \tag{3.9}$$

where $B_n(x)$ is the nth Bernoulli polynomial. Thus, (3.8) yields the formula

$$B_{2m+1}([A]) = A^{2m+1} + o(1) \qquad (C) \qquad \text{as } A \to \infty. \tag{3.10}$$

This formula is hard to prove by using other methods, even when $m = 1$.

We now give a further extension of the Poisson summation formula that holds when $\phi(0)$ is not defined. Let $\sigma_1(x) = \sigma(x) - \delta(x)$, so that

$$\sigma_1(x) = \sum_{k=-\infty}^{\infty}{}' \, \delta(x-k) - 1. \tag{3.11}$$

Then

$$\hat{\sigma}_1(x) = \sum_{k=-\infty}^{\infty}{}' \exp(2\pi ikx) - \delta(x). \tag{3.12}$$

But

$$\begin{aligned}\langle \delta(x), \phi(x)\rangle &= \left\langle \widehat{\left(\frac{1}{2\pi}\right)}, \phi(x) \right\rangle \\ &= \frac{1}{2\pi}\langle 1, \hat{\phi}(u)\rangle \\ &= \frac{1}{2\pi}\int_{-\infty}^{\infty} \hat{\phi}(u)\, du\end{aligned}$$

if $\phi \in \mathscr{S}$, so that

$$\langle \sigma_1(x), \phi(x)\rangle = \langle \sigma_1(x), \hat{\hat{\phi}}(2\pi x)\rangle, \tag{3.13}$$

and we obtain the extended Poisson summation formula

$$\begin{aligned}&\sum_{k=-\infty}^{\infty}{}' \phi(k) - \int_{-\infty}^{\infty} \phi(x)\, dx \\ &\quad = \sum_{k=-\infty}^{\infty}{}' \hat{\hat{\phi}}(2\pi k) - \int_{-\infty}^{\infty} \hat{\hat{\phi}}(2\pi x)\, dx \qquad (C).\end{aligned} \tag{3.14}$$

This formula certainly holds if $\phi \in \mathscr{K}$, but, more generally, it holds if $\phi \in \mathscr{K} + \hat{\mathscr{K}}$, that is, if $\phi = \phi_1 + \hat{\phi}_2$, with $\phi_1, \phi_2 \in \mathscr{K}$. The elements of

$\mathscr{K} + \hat{\mathscr{K}}$ are smooth away from the origin and at ∞ show the behavior of the elements of the space $\mathscr{K}$. Observe that in (3.14) both sides are to be understood in the Cesàro sense.

Before we give examples of the use of (3.14), it is worthwhile to summarize our results.

THEOREM 1. *Let $\phi \in \mathscr{K} + \hat{\mathscr{K}}$. Then the expression*

$$\sum_{1 \le |k| \le A} \phi(k) - \langle \phi(x), \chi_{(-A,A)}(x) \rangle, \tag{3.15}$$

where χ_E denotes the characteristic function of a set E, has a limit in the Cesàro sense as $A \to \infty$. Denote that limit as $\sum'^{\infty}_{k=-\infty} \phi(k) - \int_{-\infty}^{\infty} \phi(x)\, dx$. Then

$$\sum_{k=-\infty}^{\infty}{}' \phi(k) - \int_{-\infty}^{\infty} \phi(x)\, dx = \sum_{k=-\infty}^{\infty}{}' \hat{\phi}(2\pi k) - \int_{-\infty}^{\infty} \hat{\phi}(2\pi x)\, dx \qquad (C), \tag{3.16}$$

where the right-hand side is also understood as a Cesàro limit that exists because $\hat{\phi}$ also belongs to $\mathscr{K} + \hat{\mathscr{K}}$. ∎

Notice that when ϕ is even so is $\hat{\phi}$, and thus (3.1) can be written as

$$\sum_{k=1}^{\infty} \phi(k) - \int_0^{\infty} \phi(x)\, dx = \sum_{k=1}^{\infty} \hat{\phi}(2\pi k) - \int_0^{\infty} \hat{\phi}(2\pi x)\, dx \qquad (C), \tag{3.17}$$

where $\int_0^A \phi(x)\, dx = \frac{1}{2} \int_{-A}^{A} \phi(x)\, dx$ if ϕ is even.

There are two facts very important about the computation of $\sum'^{\infty}_{k=-\infty} \phi(k) - \int_{-\infty}^{\infty} \phi(x)$ if $\phi \in \mathscr{K} + \hat{\mathscr{K}}$. First, the elements of $\mathscr{K} + \hat{\mathscr{K}}$ are smooth away from the origin, and thus the evaluation $\sum_{1 \le |k| \le A} \phi(k) - \langle \phi(x), \chi_{(-A,A)}(x) \rangle$ is well defined for any $A > 0$.

Second, the elements of $\mathscr{K} + \hat{\mathscr{K}}$ are distributions of $\mathscr{S}'(\mathbb{R})$. They are not smooth functions in $\mathbb{R} \setminus \{0\}$ that need to be regularized as elements of $\mathscr{S}'(\mathbb{R})$; they are regularized already. For instance, $\phi_1(x) = x_+^{-3/2}$ is an element of $\mathscr{K} + \hat{\mathscr{K}}$, and so is $\phi_2(x) = x_+^{-3/2} + \delta(x)$. Both distributions are regularizations of the function ψ defined in $\mathbb{R} \setminus \{0\}$ by $\psi(x) = x^{-3/2}$, $x > 0$, $\psi(x) = 0$, $x < 0$. But ψ is not an element of $\mathscr{K} + \hat{\mathscr{K}}$ and thus the evaluation $\langle \psi(x), \chi_{(-A,A)}(x) \rangle$ does not make sense. The evaluations $\langle \phi_i(x), \chi_{(-A,A)}(x) \rangle$, $i = 1, 2$, are well defined. In summary, (3.16) can be applied to ϕ_1 or ϕ_2, but not to ψ.

Let us present some illustrations.

EXAMPLE 3. Let us consider the Riemann zeta function $\zeta(s)$, $s \neq 1$, defined as

$$\zeta(s) = \sum_{k=1}^{\infty} \frac{1}{k^s}, \qquad \operatorname{Re} s > 1, \tag{3.18}$$

and by analytic continuation when $\operatorname{Re} s \leq 1$, $s \neq 1$. One of the starting points in its study is the so-called functional equation [1]

$$\zeta(s) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{\pi s}{2}\zeta(1-s). \tag{3.19}$$

This equation can be proved in a variety of ways [22], but perhaps the most direct way would be by using the Poisson formula with $\phi(x) = \frac{1}{2}|x|^{-s}$, $s \in \mathbb{C} \setminus \mathbb{Z}$, so that [15, p. 158] $\hat{\phi}(u) = \Gamma(1-s)\sin(s\pi/2)|u|^{s-1}$, since

$$\sum_{k=1}^{\infty} \phi(k) = \zeta(s), \tag{3.20a}$$

$$\sum_{k=1}^{\infty} \hat{\phi}(2\pi k) = 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{s\pi}{2}\zeta(1-s), \tag{3.20b}$$

when the series converge. Unfortunately, there is no single value of s for which both series converge: (3.20a) holds if $\operatorname{Re} s > 1$, while (3.20b) holds if $\operatorname{Re} s < 0$.

On the other hand, (3.16) can be applied for any value of $s \in \mathbb{C} \setminus \mathbb{Z}$ and [4]

$$\sum_{k=-\infty}^{\infty}{}' \phi(k) - \int_{-\infty}^{\infty} \phi(x)\,dx = \zeta(s), \qquad s \in \mathbb{C}/\mathbb{Z}, \tag{3.21a}$$

$$\sum_{k=-\infty}^{\infty}{}' \hat{\phi}(2\pi k) - \int_{-\infty}^{\infty} \hat{\phi}(2\pi x)\,dx$$
$$= 2(2\pi)^{s-1}\Gamma(1-s)\sin\frac{s\pi}{2}\zeta(1-s), \qquad s \in \mathbb{C} \setminus \mathbb{Z}, \tag{3.21b}$$

and the functional equation (3.19) follows immediately.

EXAMPLE 4. Consider the generalized function $f(x) = \mathrm{F.p.}(1/|x|)$, the finite part of $1/|x|$. Clearly, $f \in \mathscr{K} + \hat{\mathscr{K}}$. Let $g = \hat{f}$. Then

$$g'(u) = \frac{d}{du}\left(\mathscr{F}\left\{\mathrm{F.p.}\left(\frac{1}{|x|}\right); u\right\}\right) = \mathscr{F}\{i \operatorname{sgn} x; u\} = -\frac{2}{u},$$

and thus

$$g(u) = a - 2\ln|u|, \tag{3.22}$$

where a is given as [15, p. 152]

$$a = \frac{1}{2}\,\mathrm{F.p.}\int_0^\infty \frac{\cos y}{y}\,dy = \frac{1}{2}\left[\int_0^1 \frac{\cos y - 1}{y}\,dy + \int_1^\infty \frac{\cos y}{y}\,dy\right]. \tag{3.23}$$

We shall now obtain another representation for a by using the modified Poisson summation formula (3.16). Actually, since f and g are even, we may use (3.17), namely,

$$\sum_{k=1}^{\infty} f(k) - \mathrm{F.p.}\int_0^\infty f(x)\,dx$$
$$= \sum_{k=1}^{\infty} g(2\pi k) - \mathrm{F.p.}\int_0^\infty g(2\pi x)\,dx \qquad (C). \tag{3.24}$$

The left-hand side is the limit in the Cesàro sense of

$$\sum_{k=1}^{[A]} \frac{1}{k} - \mathrm{F.p.}\int_0^A \frac{dx}{x} = \ln[A] + \gamma + O\left(\frac{1}{A}\right) - \ln A$$
$$= \gamma + o(1),$$

so that it equals γ, Euler's constant. The right-hand side is the Cesàro limit

$$\lim_{A\to\infty}\left[\sum_{k=1}^{[A]} (a - 2\ln 2\pi k) - \int_0^A (a - 2\ln 2\pi x)\,dx\right].$$

But

$$\lim_{A\to\infty}\sum_{k=1}^{[A]} a - \int_0^A a\,dx = a \lim_{A\to\infty}([A] - A) = -\frac{a}{2} \qquad (C),$$

while

$$\sum_{k=1}^{[A]} \ln 2\pi k - \int_0^A \ln 2\pi x\,dx = \frac{-\ln 2\pi}{2} + \ln[A]! - A\ln A + A + o(1)$$
$$= \frac{-\ln 2\pi}{2} + \left([A] + \frac{1}{2}\right)\ln[A] - [A]$$
$$+ \ln\sqrt{2\pi} - A\ln A + A + o(1)$$
$$= A\ln\frac{[A]}{A} + (A - [A]) + o(1)$$
$$= o(1) \qquad (C).$$

Thus, the right-hand side of (3.24) equals $-a/2$. Hence, $\gamma = -a/2$ or $a = -2\gamma$, and we obtain

$$g(u) = \mathscr{F}\left\{ \frac{1}{|x|}; u \right\} = -2\gamma - 2\ln|u|. \tag{3.25}$$

4. A DISTRIBUTIONAL EXTENDED FORMULA

The introduction of a parameter in the Poisson formula (1.1) gives the formula (1.2) and this formula, in turn, can be interpreted in the distributional sense. In this section we show how we can introduce a parameter in the extended summation formula (3.16) to obtain a distributional summation formula. As we show this extended distributional formula holds for *all* tempered distributions.

Let us start with some general results. Let $g \in \mathscr{K}'$ and $f \in \mathscr{S}'$. Then in several cases one may define a distribution $F \in \mathscr{S}'$ by

$$F(\lambda) = \langle g(x), f(\lambda x) \rangle. \tag{4.1}$$

The interpretation of (4.1) is observed by duality, that is, if $\phi \in \mathscr{S}$,

$$\langle F(\lambda), \phi(\lambda) \rangle = \langle g(x), \Phi(x) \rangle, \tag{4.2}$$

where

$$\Phi(x) = \langle f(\lambda x), \phi(\lambda) \rangle. \tag{4.3}$$

In general, the evaluation in the right-hand side of (4.2) cannot be performed. Indeed, Φ is smooth in $\mathbb{R} \setminus \{0\}$ and at ∞ satisfies the requirements of the elements of $\mathscr{K}$, namely, there exists $q \in \mathbb{R}$ such that $\Phi^{(j)}(x) = O(|x|^{q-j})$ as $|x| \to \infty$. But, in general, Φ is not smooth at $x = 0$.

There are two cases [6] when (4.2) can be performed for all $\phi \in \mathscr{S}$. The first is when $0 \notin \operatorname{supp} g$: in that case the lack of smoothness of Φ at the origin becomes irrelevant. The second is when all the values $f^{(n)}(0)$, $n = 0, 1, 2, \ldots,$ exist in the distributional sense of Lojasiewicz [16], because in this case ϕ is smooth at $x = 0$ for all $\phi \in \mathscr{S}$, that is, $\Phi \in \mathscr{K}$ for all $\phi \in \mathscr{S}$.

Let us now extend these ideas and show how the expression

$$F(\lambda) = \langle \sigma_1(x), f(\lambda x) \rangle, \tag{4.4}$$

or, in more suggestive notation,

$$F(\lambda) = \sum_{k=-\infty}^{\infty}{}' f(\lambda k) - \int_{-\infty}^{\infty} f(\lambda x)\, dx, \tag{4.5}$$

defines a tempered distribution $F \in \mathscr{S}'(0,\infty)$, that is, a tempered distribution defined for $\lambda > 0$. Indeed, we apply the scheme (4.2)–(4.3) with $g(x) = \sigma_1(x)$: if $\phi \in \mathscr{S}(0,\infty) = \{\psi \in \mathscr{S}(\mathbb{R})\colon \operatorname{supp} \psi \subseteq (0,\infty)\}$ then

$$\langle F(\lambda), \phi(\lambda)\rangle = \langle \sigma_1(x), \Phi(x)\rangle, \tag{4.6}$$

where

$$\Phi(x) = \langle f(\lambda x), \phi(\lambda)\rangle. \tag{4.7}$$

Observe that (4.6) is valid so long as $\Phi \in \mathscr{K} + \hat{\mathscr{K}}$. Thus, it suffices to see that $\Phi \in \mathscr{K} + \hat{\mathscr{K}}$ for all $f \in \mathscr{S}'(\mathbb{R})$, $\phi \in \mathscr{S}'(0,\infty)$. We now proceed to show that this is the case.

LEMMA 1. *Let $f \in \mathscr{S}'$. Then there exist $g, h \in \mathscr{K}'$ such that*

$$f = g + \hat{h}. \tag{4.8}$$

Proof. Observe that a distribution $f \in \mathscr{S}'$ belongs to $\mathscr{K}'$ if and only if the values $\hat{f}^{(n)}(0)$ of its Fourier transform exist in the distributional sense for $n = 0, 1, 2, \ldots$ [5]. Thus, we may construct g and h as follows. Let $\eta \in \mathscr{D}$ be a smooth function that satisfies $\eta(x) = 1$ for $-1 \le x \le 1$. Then take $g = \eta f$ and $\hat{h} = (1 - \eta)f$. The function g belongs to $\mathscr{K}'$ because it has compact support. On the other hand, $\operatorname{supp} \hat{h} \subseteq \mathbb{R} \setminus (-1, 1)$, and thus $\hat{h}$ is smooth at the origin and it follows that $h \in \mathscr{K}'$. ∎

LEMMA 2. *Let $f \in \mathscr{S}'(\mathbb{R})$ and $\phi \in \mathscr{S}(0,\infty)$. Then $\Phi(x) = \langle f(\lambda x), \phi(\lambda)\rangle$ belongs to $\mathscr{K} + \hat{\mathscr{K}}$.*

Proof. Write $f = g + \hat{h}$, where $g, h \in \mathscr{K}'$, and, accordingly, $\Phi = \Phi_1 + \Phi_2$, where

$$\Phi_1(x) = \langle g(\lambda x), \phi(\lambda)\rangle, \tag{4.9a}$$

$$\Phi_2(x) = \langle \hat{h}(\lambda x), \phi(\lambda)\rangle. \tag{4.9b}$$

In general, any distribution of the form of Φ, as Φ_1 and Φ_2, namely, $\Phi(x) = \langle f(\lambda x), \phi(\lambda)\rangle$ with $f \in \mathscr{S}'(\mathbb{R})$ and $\phi \in \mathscr{S}(0,\infty)$, belongs to $\mathscr{S}'(\mathbb{R})$, is smooth for $x \neq 0$ and at ∞ has the behavior of the elements of $\mathscr{K}$. Thus, it follows that $\Phi_2 \in \mathscr{K}$ because Φ_2 is smooth at $x = 0$ since the distributional point values $\hat{h}^{(n)}(0)$ exist for $n \in \mathbb{N}$. Similarly, $\Phi_1 \in \mathscr{K}$ because $\hat{\Phi}_1(n) = \langle \hat{g}(\rho_u)\phi(\rho^{-1})\rho^{-1}\rangle$ and since $\phi(\rho^{-1})\rho^{-1}$ also belongs to $\mathscr{S}(0,\infty)$ then $\hat{\Phi}_1$ is smooth at the origin. ∎

It follows that $\langle \sigma_1(x), f(\lambda x)\rangle$, $\lambda > 0$, is defined for all tempered distributions $f \in \mathscr{S}'$. This evaluation can be computed as a Cesàro limit.

LEMMA 3. *Let* $\sigma_1^A(x) = \sum_{1 \le |k| \le A} \delta(x-k) - \chi_{(-A,A)}(x)$ *for* $A > 0$. *Then* $\langle \sigma_1^A(x), f(\lambda x)\rangle$ *is a well-defined distribution of* $\mathscr{S}'(0,\infty)$ *for each* $f \in \mathscr{S}'(\mathbb{R})$ *and*

$$\lim_{A\to\infty} \langle \sigma_1^A(x), f(\lambda x)\rangle = \langle \sigma_1(x), f(\lambda x)\rangle \qquad (C) \tag{4.10}$$

in the topology of $\mathscr{S}'(0,\infty)$.

Proof. That $\langle \sigma_1^A(x), f(\lambda x)\rangle$ is well defined for $\lambda > 0$ is obtained by observing that $\langle \sigma_1^A(x), \Phi(x)\rangle$ is well defined if $\Phi \in \mathscr{K} + \hat{\mathscr{K}}$. To show (4.10), we should show that

$$\lim_{A\to\infty} \langle\langle \sigma_1^A(x), f(\lambda x)\rangle, \phi(\lambda)\rangle = \langle\langle \sigma_1(x), f(\lambda x)\rangle, \phi(\lambda)\rangle \qquad (C)$$

for each $\phi \in \mathscr{S}(0,\infty)$. But, setting $\Phi(x) = \langle f(\lambda x), \phi(\lambda)\rangle$, which belongs to $\mathscr{K} + \hat{\mathscr{K}}$, we obtain

$$\begin{aligned}
&\langle\langle \sigma_1^A(x), f(\lambda x)\rangle, \phi(\lambda)\rangle \\
&\quad = \langle \sigma_1^A(x), \Phi(x)\rangle \\
&\quad = \langle \sigma_1(x), \Phi(x)\rangle + o(1) \qquad (C) \qquad \text{as } A \to \infty \\
&\quad = \langle\langle \sigma_1(x), f(\lambda x)\rangle, \phi(\lambda)\rangle + o(1) \qquad (C) \qquad \text{as } A \to \infty,
\end{aligned}$$

as required. ∎

In more suggestive notation, this lemma can be written as

$$\begin{aligned}
&\lim_{A\to\infty} \left[\sum_{1\le|k|\le A} f(\lambda k) - \int_{-A}^{A} f(\lambda x)\,dx \right] \\
&\qquad = \sum_{k=-\infty}^{\infty}{}' f(\lambda k) - \int_{-\infty}^{\infty} f(\lambda x)\,dx \qquad (C),
\end{aligned} \tag{4.11}$$

a Cesàro limit that holds distributionally in λ for $\lambda > 0$ for any $f \in \mathscr{S}'$.

We can now give an extended distributional Poisson summation formula that holds for all tempered distributions.

THEOREM 2. *Let* $f \in \mathscr{S}'(\mathbb{R})$. *Then*

$$\begin{aligned}
&\sum_{k=-\infty}^{\infty}{}' f(\lambda k) - \int_{-\infty}^{\infty} f(\lambda x)\,dx \\
&\qquad = \frac{1}{\lambda} \sum_{k=-\infty}^{\infty}{}' \hat{f}\left(\frac{2k\pi}{\lambda}\right) - \frac{1}{\lambda}\int_{-\infty}^{\infty} \hat{f}\left(\frac{2\pi x}{\lambda}\right) dx \qquad (C)
\end{aligned} \tag{4.12}$$

distributionally in λ *for* $\lambda > 0$, *actually in the space* $\mathscr{S}'(0,\infty)$.

Proof. Let $\phi \in \mathscr{S}(0,\infty)$. Then

$$\left\langle \sum_{k=-\infty}^{\infty} f(\lambda k) - \int_{-\infty}^{\infty} f(\lambda x)\,dx, \phi(\lambda) \right\rangle = \langle\langle \sigma_1(x), f(\lambda x)\rangle, \phi(\lambda)\rangle$$
$$= \langle \sigma_1(x), \Phi(x)\rangle$$
$$= \langle \sigma_1(x), \hat{\Phi}(2\pi x)\rangle.$$

But

$$\begin{aligned}\hat{\Phi}(u) &= \mathscr{F}\{ f(\lambda x), \phi(\lambda)\rangle; u\} \\ &= \langle \mathscr{F}\{ f(\lambda x); u\}, \phi(\lambda)\rangle \\ &= \left\langle \frac{1}{\lambda}\hat{f}\left(\frac{u}{\lambda}\right), \phi(\lambda)\right\rangle,\end{aligned}$$

so that

$$\begin{aligned}&\langle \sigma_1(x), \hat{\Phi}(2\pi x)\rangle \\ &\quad = \left\langle \frac{1}{\lambda}\sum_{k=-\infty}^{\infty}{}' \hat{f}\left(\frac{2\pi k}{\lambda}\right) - \frac{1}{\lambda}\int_{-\infty}^{\infty} \hat{f}\left(\frac{2\pi x}{\lambda}\right) dx, \phi(\lambda)\right\rangle,\end{aligned}$$

and (4.12) follows. ∎

Let us start with a very simple example that clarifies the notation.

EXAMPLE 5. Let $f(x) = \delta(x)$. Then [15, p. 51] $\delta(\lambda x) = (1/\lambda)\delta(x) + (1/x)\delta(\lambda)$ in $\mathbb{R}^2 \setminus \{(0,0)\}$. Thus, if $\phi \in \mathscr{S}(0,\infty)$, then

$$\Phi(x) = \langle \delta(\lambda x), \phi(\lambda)\rangle = \left(\int_{-\infty}^{\infty} \phi(\lambda)/\lambda\, d\lambda\right) \delta(x),$$

since $\langle \delta(\lambda), \phi(\lambda)\rangle = 0$. Then

$$\begin{aligned}&\langle\langle \sigma_1^A(x), \delta(\lambda x)\rangle, \phi(x)\rangle \\ &\quad = \langle \sigma_1^A(x), \Phi(x)\rangle \\ &\quad = \left\langle \sigma_1^A(x), \int_{-\infty}^{\infty} \frac{\phi(\lambda)\,d\lambda}{\lambda}\,\delta(x)\right\rangle \\ &\quad = \int_{-\infty}^{\infty} \frac{\phi(\lambda)\,d\lambda}{\lambda}\left(\sum_{1\le k\le A} \delta(k) - \int_{-A}^{A} \delta(x)\,dx\right) \\ &\quad = -\int_{-\infty}^{\infty} \frac{\phi(\lambda)\,d\lambda}{\lambda}\end{aligned}$$

for any $A > 0$. Thus,

$$\sum_{k=-\infty}^{\infty}{}' \delta(\lambda k) - \int_{-\infty}^{\infty} \delta(\lambda x)\, dx = -\frac{1}{\lambda}, \qquad \lambda > 0. \tag{4.13}$$

On the other hand, $\hat{f}(x) = 1$, and thus

$$\left\langle \sigma_1^A(x), \frac{1}{\lambda}\left(\frac{2\pi x}{\lambda}\right)\right\rangle = \frac{1}{\lambda}\left[\sum_{1 \le k \le A} 1 - \int_{-A}^{A} dx\right]$$
$$= \frac{1}{\lambda}(2[A] - A) = \frac{-2\{A\}}{\lambda},$$

and since $\{A\} = 1/2 + o(1)\ (C)$ as $A \to \infty$, we obtain

$$\left\langle \sigma_1(x), \frac{1}{\lambda}\left(\frac{2\pi x}{\lambda}\right)\right\rangle = \frac{-1}{\lambda}, \qquad \lambda > 0, \tag{4.14}$$

as expected.

We shall consider a few more interesting illustrations. But before we do that we need some extra machinery. First, observe that when f is even the Poisson summation formula (4.12) can be written as

$$\sum_{k=1}^{\infty} f(\lambda k) - \int_0^{\infty} f(\lambda x)\, dx$$
$$= \frac{1}{\lambda}\sum_{k=1}^{\infty} \hat{f}\left(\frac{2\pi k}{\lambda}\right) - \frac{1}{\lambda}\int_0^{\infty} \hat{f}\left(\frac{2\pi x}{\lambda}\right) dx \qquad (C) \tag{4.15}$$

distributionally for $\lambda > 0$, where both sides are to be computed in the Cesàro sense. Recall that $\int_0^A f(\lambda x)\, dx = \frac{1}{2}\int_{-A}^{A} f(\lambda x)\, dx$ is the convention used, because f is even.

The power of the Poisson summation formula is enhanced by the knowledge of the asymptotic behavior of the function $F(\lambda) = \sum_{k=1}^{\infty} f(k\lambda) - \int_0^{\infty} f(\lambda x)\, dx$ as $\lambda \to 0^+$. According to a result that was given by Ramanujan [20] and proved recently [2, 4], if ψ that is smooth in $(0, \infty)$ has the behavior of the type of the space $\mathscr{K}$ at ∞ while at the origin has the asymptotic expansion

$$\psi(x) \sim a_1 x^{\alpha_1} + a_2 x^{\alpha_2} + a_3 x^{\alpha_3} + \cdots \qquad \text{as } x \to 0^+,$$

where Re $\alpha_j \uparrow \infty$ and $\alpha_j \neq -1$ for all j, then

$$\sum_1^{\infty} \psi(k\epsilon) - \int_0^{\infty} \psi(\epsilon x)\, dx$$
$$\sim a_1 \zeta(-\alpha_1) \epsilon^{\alpha_1} + a_2 \zeta(-\alpha_2) \epsilon^{\alpha_2} + a_3 \zeta(-\alpha_3) \epsilon^{\alpha_3} + \cdots, \quad (4.16)$$

where $\zeta(s)$ is the Riemann zeta function. The expansion when $\alpha_m = -1$ for some index m is also known [4], but will not be needed presently.

It turns out that when the smooth function ψ is replaced by a tempered distribution f then an expansion completely analogous to (4.16) holds, but in the *distributional sense*. Indeed, we say that a distribution $f \in \mathscr{S}'$ has the average or distributional expansion $\sum_{j=0}^{\infty} a_j x_+^{\alpha_j}$ as $x \to 0^+$ if f has the parametric behavior

$$f(\epsilon x) \sim \sum_{j=0}^{\infty} a_j x_+^{\alpha_j} \epsilon^{\alpha_j} \qquad \text{as } \epsilon \to 0^+, \quad (4.17)$$

in the sense that

$$\langle f(\epsilon x), \phi(x) \rangle \sim \sum_{j=0}^{\infty} a_j \langle x_+^{\alpha_j}, \phi(x) \rangle \epsilon^{\alpha_j} \qquad \text{as } \epsilon \to 0^+, \quad (4.18)$$

for each $\phi \in \mathscr{S}(0, \infty)$. Then the behavior of $F(\lambda) = \sum_{k=1}^{\infty} f(k\lambda) - \int_0^{\infty} f(\lambda x)\, dx$ at the origin can be obtained by studying the behavior of $\langle F(\epsilon\lambda), \phi(\lambda) \rangle$ as $\epsilon \to 0^+$ where $\phi \in \mathscr{S}(0, \infty)$. But if $\Phi(x) = \langle f(\lambda x), \phi(\lambda) \rangle$, then

$$\langle F(\epsilon\lambda), \phi(\lambda) \rangle = \sum_{k=1}^{\infty} \Phi(k\epsilon) - \int_0^{\infty} \Phi(\epsilon x)\, dx$$
$$\sim \sum_{j=0}^{\infty} a_j \zeta(-\alpha_j) \langle \lambda_+^{\alpha_j}, \phi(\lambda) \rangle \epsilon^{\alpha_j} \qquad \text{as } \epsilon \to 0^+,$$

because the Ramanujan asymptotic series (4.16) is applicable when $\psi = \Phi$, and Φ has the expansion

$$\Phi(x) \sim \sum_{j=0}^{\infty} a_j \langle \lambda_+^{\alpha_j}, \phi(\lambda) \rangle x^{\alpha_j} \qquad \text{as } x \to 0^+.$$

It follows that

$$F(\epsilon\lambda) \sim \sum_{j=0}^{\infty} a_j \zeta(-\alpha_j) \lambda_+^{\alpha_j} \epsilon^{\alpha_j} \qquad \text{as } \epsilon \to 0^+,$$

and thus

$$\sum_{k=1}^{\infty} f(k\lambda) - \int_0^{\infty} f(\lambda x)\,dx$$
$$\sim \sum_{j=0}^{\infty} a_j \zeta(-\alpha_j)\lambda_+^{\alpha_j} \qquad \text{distributionally as } \lambda \to 0^+. \tag{4.19}$$

EXAMPLE 6. Let us consider the function $\Lambda(a)$ defined by the nonharmonic Fourier series

$$\Lambda(a) = \sum_{n=1}^{\infty} \sin a\sqrt{n}\,, \qquad a > 0. \tag{4.20}$$

This series is divergent for each $a > 0$, but $\Lambda(a)$ can be understood in the distributional sense by observing that if $f(x) = \sin|x|^{1/2}$, then

$$\Lambda(\alpha^{1/2}) = \tfrac{1}{2}\langle \sigma_1(x), f(\alpha x)\rangle,$$

since $\int_{-\infty}^{\infty} \sin \alpha^{1/2}|x|^{1/2}\,dx = 0$ (C) for any $\alpha > 0$.

An alternative expression for $\Lambda(a)$ is obtained by applying the extended distributional Poisson formula (4.15) with $f(x) = \sin|x|^{1/2}$ and $\lambda = a^{1/2}$. Since $\hat{f}(u) = -\sqrt{\pi}|u|^{-3/2}\cos(1/4|u| + \pi/4)$ and since $\int_{-\infty}^{\infty} \hat{f}(u)\,du = 0$, we obtain

$$\Lambda(a) = \frac{-a}{2\sqrt{2}\,\pi} \sum_{k=1}^{\infty} k^{-3/2} \cos\left(\frac{a^2}{8\pi k} + \frac{\pi}{4}\right), \qquad a > 0. \tag{4.21}$$

Observe that the series on the right-hand side converges for $a > 0$ but, not only that, it converges for all $a \in \mathbb{C}$. It follows that the function $\Lambda(a)$, which was initially defined for $a > 0$, admits a continuation to $\mathbb{C}$ as an entire function, which we continue to denote as $\Lambda(a)$.

The Taylor expansion for $\Lambda(a)$ can be obtained by using (3.18). Indeed, $\sin\sqrt{x} \sim x^{1/2}/3! + x^{5/2}/5! - \cdots$ as $x \to 0^+$ and thus

$$\begin{aligned}
\Lambda(a) &= \sum_{k=1}^{\infty} \sin a\sqrt{k} \\
&= \sum_{k=1}^{\infty} \sin\sqrt{a^2 k} \\
&\sim \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(-j - 1/2)\, a_+^{2(j+1/2)}}{(2j+1)!}
\end{aligned}$$

or

$$\Lambda(a) \sim \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(-j-1/2) a_+^{2j+1}}{(2j+1)!} \qquad \text{as } a \to 0^+ \text{ distributionally}. \tag{4.22}$$

But $\Lambda(a)$ is an entire function and so

$$\Lambda(a) = \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(-j-1/2) a_+^{2j+1}}{(2j+1)!}, \qquad a \in \mathbb{C}. \tag{4.23}$$

We now consider some related nonharmonic Fourier series.

EXAMPLE 7. The function Λ_q, $q \in \mathbb{N}$, defined by

$$\Lambda_q(a) = \sum_{k=1}^{\infty} k^q \sin a\sqrt{k}, \qquad a > 0, \tag{4.24}$$

can be analyzed by the procedures of the previous example. However, it is easier to observe that $\Lambda_0 = \Lambda$ and that $\Lambda_q = (-1)^q \Lambda_0^{(2q)}$. It follows that $\Lambda_q(a)$ extends to an entire function and using (4.23) that

$$\Lambda_q(a) = \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(-j-q-1/2) a_+^{2j+1}}{(2j+1)!}, \qquad a \in \mathbb{C}. \tag{4.25}$$

The analysis can be extended to negative values of q. For instance,

$$\Lambda_{-1}(a) = \sum_{k=1}^{\infty} \frac{\sin a\sqrt{k}}{k}, \qquad a > 0, \tag{4.26}$$

is a function whose second derivative is $-\Lambda_0(a)$ and, consequently, can be extended to all $a \in \mathbb{C}$ as an entire function. Its Taylor series can be obtained from the generalized Ramanujan expansion (4.19):

$$\begin{aligned}
&\sum_{k=1}^{\infty} \frac{\sin a\sqrt{k}}{k} \\
&\quad = \left(\sum_{k=1}^{\infty} \frac{\sin a\sqrt{k}}{k} - \int_0^{\infty} \frac{\sin a\sqrt{x}}{x}\, dx \right) + \pi \\
&\quad = a^2 \left(\sum_{k=1}^{\infty} \frac{\sin\sqrt{a^2 k}}{ak} - \int_0^{\infty} \frac{\sin\sqrt{a^2 k}}{a^2 k}\, dx \right) + \pi \\
&\quad \sim a^2 \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(1/2 - j) a_+^{2j-1}}{(2j+1)!} + \pi \\
&\quad \sim \pi + \sum_{j=0} \frac{(-1)^j \zeta(1/2 - j) a_+^{2j+1}}{(2j+1)!}, \qquad a \to 0^+ \text{ distributionally,}
\end{aligned}$$

so that

$$\Lambda_{-1}(a) = \pi + \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(1/2 - j) a_+^{2j+1}}{(2j+1)!}, \qquad a \in \mathbb{C}. \tag{4.27}$$

This formula was found by Boersma [3] when solving a problem proposed by Glasser [11].

It is interesting to observe that when $a < 0$ the nonharmonic series converges, but not to $\Lambda_{-1}(a)$, namely,

$$\begin{aligned}
\sum_{k=1}^{\infty} \frac{\sin a\sqrt{k}}{k} &= -\Lambda_1(|a|) \\
&= -\pi - \sum_{j=0}^{\infty} \frac{(-1)^j \zeta(1/2 - j) |a|^{2j+1}}{(2j+1)!},
\end{aligned}$$

and thus

$$\sum_{k=1}^{\infty} \frac{\sin a\sqrt{k}}{k} = -2\pi + \Lambda_{-1}(a), \qquad a < 0. \tag{4.28}$$

The nonharmonic Fourier series $\sum_{k=1}^{\infty} \sin a\sqrt{k}\,/k$ is not Cesàro summable when $a \in \mathbb{C} \setminus \mathbb{R}$.

5. MULTIDIMENSIONAL FORMULAS

We shall now present multidimensional versions of the modified Poisson summation formulas.

Throughout this section we shall work in the space $\mathbb{R}^d$. We denote by S the unit sphere, $S = \{\mathbf{x} \in \mathbb{R}^d: |\mathbf{x}| = 1\}$, and by B the unit ball, $B = \{\mathbf{x} \in \mathbb{R}^d: |\mathbf{x}| < 1\}$.

Our first concern is the notion of Cesàro summability by spherical means. Let $f \in \mathscr{K}'(\mathbb{R}^d)$ and $\phi \in \mathscr{K}(\mathbb{R}^d)$. We show that the evaluation $\langle f(\mathbf{x}), \phi(\mathbf{x})\rangle$ can be computed as a Cesàro limit of spherical means, so that, in particular, if f is locally integrable, then

$$\lim_{A \to \infty} \int_{|\mathbf{x}| \le A} f(\mathbf{x})\,\phi(\mathbf{x})\,d\mathbf{x} = \langle f(\mathbf{x}), \phi(\mathbf{x})\rangle \qquad (C). \tag{5.1}$$

It is enough to consider the case when $\phi = 1$ since

$$\langle f(\mathbf{x}), \phi(\mathbf{x})\rangle = \langle \phi(\mathbf{x}) f(\mathbf{x}), 1\rangle \qquad \text{and} \qquad \phi f \in \mathscr{K}'.$$

As we now show, by using polar coordinates, we may reduce the problem to the one-dimensional case. Indeed, if $\psi \in \mathscr{D}(S)$ is a smooth function defined on the unit sphere, then we may define the spherical-mean-generalized function $\langle f(r\boldsymbol{\omega}), \psi(\boldsymbol{\omega})\rangle r_+^{d-1}$ which is a distribution of the one-variable r defined by

$$\langle\langle f(r\boldsymbol{\omega}), \psi(\boldsymbol{\omega})\rangle r_+^{d-1}, \rho(r)\rangle = \langle f(\mathbf{x}), R(\mathbf{x})\rangle, \tag{5.2}$$

where $R(\mathbf{x}) = \rho(|\mathbf{x}|)\psi(|\mathbf{x}|^{-1}\mathbf{x})$. The spherical-mean distribution $\langle f(r\boldsymbol{\omega}), \psi(\boldsymbol{\omega})\rangle r_+^{d-1}$ is a distribution of $\mathscr{D}'(\mathbb{R} \setminus \{0\})$ but not of $\mathscr{D}'(\mathbb{R})$ because R might not be smooth at the origin for some $\rho \in \mathscr{D}(\mathbb{R})$. However, $\langle f(r\boldsymbol{\omega})\psi(\boldsymbol{\omega})\rangle r_+^{d-1}$ admits extensions to $\mathscr{D}'(\mathbb{R})$ that satisfy $\langle F(r), \rho(r)\rangle = \langle f(\mathbf{x}), R(\mathbf{x})\rangle$ whenever $R \in \mathscr{D}(\mathbb{R}^d)$.

We now use our assumption that $f \in \mathscr{K}'(\mathbb{R}^d)$ to obtain that any extension $F \in \mathscr{D}(\mathbb{R})$ of the spherical-mean distribution $\langle f(r\boldsymbol{\omega}), \psi(\boldsymbol{\omega})\rangle r_+^{d-1}$ belongs to $\mathscr{K}'(\mathbb{R})$. Therefore, the one-dimensional theory shows that $\langle F(r), \rho(r)\rangle$ can be evaluated as a Cesàro limit for any $\rho \in \mathscr{K}(\mathbb{R})$, which gives a corresponding result for the evaluation $\langle f(\mathbf{x}), R(\mathbf{x})\rangle$. If we apply this to the case when $R = 1$, that is, $\psi = 1$, $\rho = 1$, we find that the evaluation $\langle f(\mathbf{x}), 1\rangle$ can be computed as a Cesàro limit of spherical means.

Summarizing, we have

THEOREM 3. *Let $f \in \mathscr{K}'(\mathbb{R}^d)$ and $\phi \in \mathscr{K}(\mathbb{R}^d)$. Then the evaluation $\langle f(\mathbf{x}), \phi(\mathbf{x})\rangle$ can be computed as the Cesàro limit of the spherical means, in the sense that if $G(r)$ is the primitive with support bounded on the left of any extension of $\langle f(r\boldsymbol{\omega}), \phi(r\boldsymbol{\omega})\rangle_S r_+^{d-1}$, then $G(r)$ has a limit in the Cesàro sense as $r \to \infty$ and $\lim_{r\to\infty} G(r) = \langle f(\mathbf{x}), \phi(\mathbf{x})\rangle$ (C).*

We now take

$$\sigma(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d} \delta(\mathbf{x}-\mathbf{k}) - 1. \tag{5.3}$$

Then $\sigma \in \mathscr{K}'(\mathbb{R}^d)$ [9]. If $\phi \in \mathscr{K}(\mathbb{R}^d)$, then the evaluation $\langle \sigma(\mathbf{x}), \phi(\mathbf{x})\rangle$ gives a precise meaning to the expression $\sum_{\mathbf{k}\in\mathbb{Z}^d}\phi(\mathbf{k}) - \int_{\mathbb{R}^d}\phi(\mathbf{x})\,d\mathbf{x}$, which, in general, is the difference of a divergent series and a divergent integral. By using the theorem, it follows that $\langle \sigma(\mathbf{x}), \phi(\mathbf{x})\rangle$ can be evaluated as the Cesàro limit of the spherical partial sums $\sum_{|\mathbf{k}|\le A}\phi(\mathbf{k}) - \int_{|\mathbf{x}|\le A}\phi(\mathbf{x})\,d\mathbf{x}$ as $A\to\infty$.

We can now give our first modified Poisson summation formula.

THEOREM 4. *Let $\phi \in \mathscr{K}(\mathbb{R}^d)$. Then*

$$\sum_{\mathbf{k}\in\mathbb{Z}^d}\phi(\mathbf{k}) - \int_{\mathbb{R}^d}\phi(\mathbf{x})\,d\mathbf{x} = \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' \,\hat{\phi}(2\pi\mathbf{k}) \qquad (C), \tag{5.4}$$

where the left-hand side is the Cesàro limit of the spherical partial sums, while the right-hand side is an absolutely summable series.

Proof. The proof follows by observing that the Fourier expansion $\sigma(\mathbf{x}) = \sum_{\mathbf{k}\in\mathbb{Z}^d}\exp(2\pi i\mathbf{k}\cdot\mathbf{x})$ holds in $\mathscr{K}'(\mathbb{R}^d)$. ∎

Even the following simplest example of (5.4) is interesting.

EXAMPLE 8. Take $\phi(\mathbf{x}) = 1$ so that $\hat{\phi}(\mathbf{u}) = (2\pi)^d\,\delta(\mathbf{u})$. Then the right-hand side of (5.4) vanishes. Hence,

$$\lim_{A\to\infty}\left[\sum_{|\mathbf{k}|\le A} 1 - \int_{|\mathbf{x}|\le A} d\mathbf{x}\right] = 0 \qquad (C). \tag{5.5}$$

Let $N(A) = \sum_{|\mathbf{k}|\le A} 1$, the number of lattice points of $\mathbb{Z}^d$ in a circle of radius A. Then (5.5) can be rewritten as

$$N(A) = \frac{\pi^{d/2}A^d}{\Gamma(d/2+1)} + o(1) \qquad (C) \qquad \text{as } A\to\infty. \tag{5.6}$$

This is an average relation, (5.6) is not true in the ordinary sense. Indeed,

$$\liminf_{A\to\infty}\left|N(A) - \frac{\pi^{d/2}A^d}{\Gamma(d/2+1)}\right| A^{1-d} > 0,$$

as can be easily verified.

As in the one-dimensional case, a stronger modified Poisson summation formula is valid in $\mathscr{K} + \hat{\mathscr{K}}$. It follows by considering the distribution $\sigma_1(\mathbf{x}) = \sigma(\mathbf{x}) - \delta(\mathbf{x})$.

THEOREM 5. *Let $\phi \in \mathscr{K}(\mathbb{R}^d) + \hat{\mathscr{K}}(\mathbb{R}^d)$. Then*

$$\sum_{\mathbf{x}\in\mathbb{Z}^d}{}' \phi(\mathbf{k}) - \int_{\mathbb{R}^d} \phi(\mathbf{x})\, d\mathbf{x}$$

$$= \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' \hat{\phi}(2\pi\mathbf{k}) - \int_{\mathbb{R}^d} \hat{\phi}(2\pi\mathbf{x})\, d\mathbf{x} \qquad (C), \tag{5.7}$$

when both sides are computed as Cesàro limits of the spherical partial sums.

We shall now show how our methods yield a rather simple analysis of the zeta function.

EXAMPLE 9. Consider the function $V(s)$ initially defined as

$$V(s) = \sum_{\mathbf{k}\in\mathbb{Z}^d} |\mathbf{k}|^s, \qquad \operatorname{Re} s < -d. \tag{5.8}$$

Let $\phi_0 \in \mathscr{K}(\mathbb{R}^d)$ be a fixed function that satisfies $\phi_0(\mathbf{x}) = 0$, $|\mathbf{x}| < 1/4$, and $\phi_0(\mathbf{x}) = 1$, $|\mathbf{x}| > 3/4$. Let $\phi_s(\mathbf{x}) = |\mathbf{x}|^s\phi_0(\mathbf{x})$. Then $\phi_s \in \mathscr{K}(\mathbb{R}^d)$ for any $s \in \mathbb{C}$. Let $\boldsymbol{\sigma}(\mathbf{x}) = \sigma_1(\mathbf{x}) - \chi_B(\mathbf{x})$, where χ_B is the characteristic function of the unit ball B. Since $\boldsymbol{\sigma} \in \mathscr{K}'(\mathbb{R}^d)$, it follows that $Z(s) = \langle \boldsymbol{\sigma}(\mathbf{x}), \phi_s(\mathbf{x})\rangle$ is an entire function of s. That is,

$$Z(s) = \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' |\mathbf{k}|^s - \int_{|\mathbf{x}|\geq 1} |\mathbf{x}|^s\, d\mathbf{x} \qquad (C) \tag{5.9}$$

is well defined as a Cesàro limit of spherical partial sums for any $s \in \mathbb{C}$ and defines an entire function.

When $\operatorname{Re} s < -d$ both the series and the integral are convergent. The integral can be easily computed as $\int_{|\mathbf{x}|\geq 1} |\mathbf{x}|^s\, d\mathbf{x} = -\omega d/(s+d)$, where $w_d = 2\pi^{d/2}/\Gamma(d/2)$ is the area of the unit sphere S of $\mathbb{R}^d$. Thus, from (5.8) and (5.9) it follows that

$$V(s) = Z(s) + \frac{\omega_d}{s+d} \tag{5.10}$$

for $\operatorname{Re} s < -d$. But since $Z(s)$ is entire, we immediately obtain that $V(s)$ admits an analytic continuation to $\mathbb{C} \setminus \{-d\}$ and that $s = -d$ is a simple pole with residue ω_d.

An even simpler representation of $V(s)$ is obtained by observing the value of the finite part integral [7]

$$\text{F.p.}\int_{|\mathbf{x}|\leq 1} |\mathbf{x}|^s \, d\mathbf{x} = \frac{\omega_d}{s+d}, \qquad s \neq -d, \tag{5.11}$$

from which we obtain

$$V(s) = \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' |\mathbf{k}|^s - \text{F.p.}\int_{\mathbb{R}_d} |\mathbf{x}|^s \, d\mathbf{x} \qquad (C), \qquad s \neq -d. \tag{5.12}$$

The finite part handles the divergence at the origin, while the Cesàro summability takes care of the divergence at ∞. Interestingly, if one interprets the always divergent integral $\int_{\mathbb{R}^d} |\mathbf{x}|^s \, d\mathbf{x}$ in the analytic continuation sense of Gelfand and Shilov [10], one gets $\int_{\mathbb{R}^d} |\mathbf{x}|^s \, d\mathbf{x} = 0$ for all $s \in \mathbb{C}$. Therefore, in a certain sense, (5.8) holds for all $s \in \mathbb{C} \setminus \{-d\}$.

We cannot set $s = -d$ in (5.12) because the left-hand side $V(s)$ is not defined if $s = -d$. However, the right-hand side exists and equals $Z(-d)$ because of (5.10) and the value $\text{F.p.}\int_{|\mathbf{x}|\leq 1} |\mathbf{x}|^{-d} \, d\mathbf{x} = 0$. The number $Z(-d)$ is the finite part value of the analytic function $V(s)$ at the pole $s = -d$, as (5.10) shows. Thus,

$$\text{F.p.} \lim_{s\to -d} V(s) = \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' |\mathbf{k}|^{-d} - \text{F.p.}\int_{\mathbb{R}^d} |\mathbf{x}|^{-d} \, d\mathbf{x} \qquad (C). \tag{5.13}$$

Alternatively, (5.12) and (5.13) can be rewritten as

$$V(s) = \lim_{A\to\infty} \left[\sum_{1\leq|\mathbf{k}|\leq A} |\mathbf{k}|^s - \frac{\omega_d A^s}{s+d} \right] \qquad (C), \qquad s \neq -d, \tag{5.14}$$

and

$$\text{F.p.} \lim_{s\to -d} V(s) = \lim_{A\to\infty} \left[\sum_{1\leq|\mathbf{k}|\leq A} |\mathbf{k}|^{-d} - \omega_d \ln A \right] \qquad (C). \tag{5.15}$$

The functional equation is obtained by using the modified Poisson summation formula (5.4) with $\phi(\mathbf{x}) = |\mathbf{x}|^s$, which belongs to $\mathscr{K}(\mathbb{R}^d) + \hat{\mathscr{K}}(\mathbb{R}^d)$. We suppose $s \notin \mathbb{Z}$ in order to avoid the consideration of the special values of the Fourier transform $\hat{\phi}(\mathbf{u})$; these values will be recovered by analytic continuation later on. Hence, we use the formula

$$\hat{\phi}(\mathbf{u}) = C_{s,d} |\mathbf{u}|^{-s-d}, \tag{5.16}$$

where

$$C_{s,d} = \frac{2^{s+d}\pi^{d/2}\Gamma((s+d)/2)}{\Gamma(-s/2)}, \tag{5.17}$$

to obtain

$$\begin{aligned} V(s) &= \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' |\mathbf{k}|^s - \mathrm{F.p.}\int_{\mathbb{R}^d} |\mathbf{x}|^s \, d\mathbf{x} \qquad (C) \\ &= C_{s,d}\left(\sum_{\mathbf{k}\in\mathbb{Z}^d}{}' |2\pi\mathbf{k}|^{-s-d} - \mathrm{F.p.}\int_{\mathbb{R}^d} |2\pi\mathbf{x}|^{-s-d} \, d\mathbf{x}\right) \qquad (C) \\ &= C_{s,d}(2\pi)^{-s-d} V(-s-d) \end{aligned}$$

or

$$V(S) = \frac{\pi^{-s-d/2}\Gamma((s+d)/2)}{\Gamma(-s/2)} V(-s-d). \tag{5.18}$$

This holds for $s \in \mathbb{C} \setminus \mathbb{Z}$ and, by analytic continuation, for $s \in \mathbb{C} \setminus \{-d, 0\}$.

There is also a distributional modified Poisson summation formula in $\mathbb{R}^d$. As in the one-dimensional case, the formula holds for all tempered distributions. The derivation is quite similar to the one-dimensional case, based on the representation $\mathscr{S}' = \mathscr{K}' + \hat{\mathscr{K}}'$, which also holds in $\mathbb{R}^d$.

THEOREM 6. *Let $f \in \mathscr{S}'(\mathbb{R}^d)$. Then both sides of the following equation exist as distributional Cesàro limits of spherical means for* $\lambda > 0$, *and we have the identity*

$$\begin{aligned} &\sum_{\mathbf{k}\in\mathbb{Z}^d}{}' f(\lambda\mathbf{k}) - \int_{\mathbb{R}^d} f(\lambda\mathbf{x}) \, d\mathbf{x} \\ &\qquad = \frac{1}{\lambda^d}\left[\sum_{\mathbf{k}\in\mathbb{Z}^d}{}' \hat{f}\left(\frac{2\pi\mathbf{k}}{\lambda}\right) - \int_{\mathbb{R}^d} \hat{f}\left(\frac{2\pi\mathbf{x}}{\lambda}\right) d\mathbf{x}\right] \qquad (C) \end{aligned} \tag{5.19}$$

in the distributional space $\mathscr{S}'(0,\infty)$.

We finish by giving a last illustration. Other interesting examples can be found in [17, 18].

EXAMPLE 10. Take $f(\mathbf{x}) = \exp(ia|x|^2)$, where as $a \in \mathbb{R} \setminus \{0\}$. Define the function $H(a)$ by

$$H(a) = \sum_{\mathbf{k}\in\mathbb{Z}^d}{}' \exp(ia|\mathbf{k}|^2) - \int_{\mathbb{R}^d} \exp(ia|\mathbf{x}|^2) \, d\mathbf{x} \qquad (C). \tag{5.20}$$

Since $\hat{f}(\mathbf{u}) = (-\pi/ia)^{d/2}\exp(-i|\mathbf{u}|^2/4a)$, we obtain from (5.19) with $a = \sqrt{\lambda}$ the functional equation

$$F(a) = \frac{(\pi i)^{d/2}}{a^d} F\left(-\frac{\pi^2}{a}\right). \tag{5.21}$$

The integral in (5.20) can be easily computed, but the series cannot be interpreted as an ordinary function even when $d = 1$, as follows from [13].

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